DETERMINATION OF THE TEMPERATURE FIELD IN PLATE WITH A DISK FOR A NONIDEAL THERMAL CONTACT BETWEEN THEM

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The problem of determining the stationary temperature field in a plate with a disk of arbitrary profile is discussed for the case of a nonideal thermal contact between the plate and the disk. Heat transfer according to Newton's law takes place between the plate and the ambient medium.

Let a disk of arbitrary configuration be soldered into a thin elastic unbounded plate. Between the plate and the disk, there is a thin intermediate layer; heat transfer, symmetrical with respect to the middle surface, takes place with the ambient medium according to Newton's law. We substitute a physical curve L with normalized thermophysical parameters for the intermediate layer. The problem of determining a generalized two-dimensional temperature field T(x, y) in such a plate reduced to the solution of the equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}-\varkappa^2 T=-\varkappa^2 t_m \tag{1}$$

for certain specific conditions at infinity and on the curve L of the joint.

The general solution to Eq. (1) is chosen in the form

$$T(x,\ y)=t(x,\ y)+t^*(x,\ y), \tag{2}$$

where t is the solution of a homogeneous equation that corresponds to Eq. (1).

We assume that at infinity $t(x, y) \to 0$ and that at the contour L, t satisfies the conditions of nonideal thermal contact [2, 3]:

$$\lambda_0\frac{\partial^2}{\partial s_0^2}(t^+ + t^-)+2\left(\lambda_1\frac{\partial t^+}{\partial n_0}-\lambda_2\frac{\partial t^-}{\partial n_0}\right)=f_1(s_0),$$

$$\lambda_0\frac{\partial^2}{\partial s_0^2}(t^+ - t^-)+6\left(\lambda_1\frac{\partial t^+}{\partial n_0}+\right.$$

$$\left.+\lambda_2\frac{\partial t^-}{\partial n_0}\right)-12h(t^+ - t^-)=f_2(s_0), \tag{3}$$

where

$$f_1(s_0)=-2\left[\lambda_0\frac{\partial^2 t^*}{\partial s_0^2}+(\lambda_1-\lambda_2)\frac{\partial t^*}{\partial n_0}\right];$$

$$f_2(s_0)=-6(\lambda_1+\lambda_2)\frac{\partial t^*}{\partial n_0} \tag{4}$$

The solution to the homogeneous equation

$$\Delta t-\varkappa^2 t=0, \tag{5}$$

that satisfies the conditions at the contour (3) and vanishes at infinity is taken in the form

$$t(x,\ y)=v(x,\ y)+u(x,\ y), \tag{6}$$

where

$$v(x,\ y)=\frac{1}{2\pi}\int_L p(s)K_0(\varkappa r)\,ds; \tag{7}$$

$$u(x,\ y)=\frac{1}{2\pi}\int_L \gamma(s)\frac{d}{dn}K_0(\varkappa r)\,ds. \tag{8}$$

The functions v(x, y) and u(x, y) are the solutions of the heat-conduction equation for a plate with heat transfer under the action of sources and dipoles having the densities p(s) and $\gamma(s)$, respectively, located on curve L. These functions are analogs of the logarithmic potentials of single and double layers, while for a thermally insulated plate ($\varkappa = 0$), they reduce to ordinary potentials.

Since the functions v(x, y) and u(x, y) possess the properties of potentials of single and double layers, we get the following expressions for their limiting values at the contour L:

$$v^\pm(s_0)=\frac{1}{2\pi}\int_L p(s)K_0(\varkappa r)\,ds,$$

$$u^\pm(s_0)=\pm\frac{1}{2}\gamma(s_0)+\frac{\varkappa}{2\pi}\int_L \gamma(s)\sin\alpha\,K_1(\varkappa r)\,ds. \tag{9}$$

Taking into consideration the properties of the functions $K_0(\varkappa r)$ and $K_1(\varkappa r)$, the limiting values of the normal derivatives and second tangential derivatives of the functions v and u can be obtained as:

$$\frac{dv^\pm}{dn_0}=\mp\frac{1}{2}p(s_0)+\frac{\varkappa}{2\pi}\int_L p(s)\sin\alpha_0\,K_1(\varkappa r)\,ds,$$

$$\frac{du^\pm}{dn_0}=\frac{\varkappa}{2\pi}\int_L \gamma'(s)\cos\alpha_0\,K_1(\varkappa r)\,ds-$$

$$-\frac{\varkappa^2}{2\pi}\int_L \gamma(s)\cos(\alpha+\alpha_0)K_0(\varkappa r)\,ds,$$

$$\frac{d^2v^\pm}{ds_0^2}=\frac{\varkappa}{2\pi}\int_L p'(s)\cos\alpha\,K_1(\varkappa r)\,ds+$$

$$+\frac{\varkappa}{2\pi}\int_L p(s)\left[2\frac{\sin^2\alpha-\sin^2\alpha_0}{r}-\right.$$

$$\left.-k\sin\alpha+k_0\sin\alpha_0\right]K_1(\varkappa r)\,ds-$$

$$-\frac{\varkappa^2}{2\pi}\int_L p(s)(\cos^2\alpha-\cos^2\alpha_0)K_0(\varkappa r)\,ds,$$

$$\frac{d^2u^{\pm}}{ds_0^2}=\pm\frac{1}{2}\gamma''(s_0)+$$

$$+\frac{\varkappa}{2\pi}\int_L\gamma'(s)\left[k_0\cos\alpha_0-\frac{\sin 2\alpha_0}{r}\right]K_1(\varkappa r)\,ds+$$

$$+\frac{\varkappa^2}{2\pi}\int_L\gamma(s)\left[\frac{\sin\alpha\cos 2\alpha_0}{r}-\right.$$

$$\left.-k_0\cos(\alpha+\alpha_0)\right]K_0(\varkappa r)\,ds+$$

$$+\frac{\varkappa^3}{2\pi}\int_L\gamma(s)\sin\alpha\cos^2\alpha_0 K_1(\varkappa r)\,ds. \qquad (10)$$

All integrals in formulas (10), which contain the expressions $\cos\alpha\, K_1(\varkappa r)$ and $\cos\alpha_0\, K_1(\varkappa r)$ should be understood in the sense of the principal value according to Cauchy.

Substituting (9) and (10), with allowance for (6), into boundary conditions (3), we get the following system of singular integrodifferential equations defining p(s) and $\gamma(s)$:

$$\lambda_0\varkappa\int_L\left\{\gamma'(s)\left[\eta\cos\alpha_0-\frac{\sin 2\alpha_0}{r}\right]+\right.$$

$$+p'(s)\cos\alpha+p(s)\left[\eta\sin\alpha_0-\right.$$

$$\left.\left.-k\sin\alpha+2\frac{\sin^2\alpha-\sin^2\alpha_0}{r}\right]\right\}K_1(\varkappa r)\,ds+$$

$$+\lambda_0\varkappa^2\int_L\left\{\gamma(s)\left[\frac{\sin\alpha}{r}\sin 2\alpha_0-\eta\cos(\alpha+\alpha_0)\right]-\right.$$

$$\left.-p(s)[\cos^2\alpha-\cos^2\alpha_0]\right\}K_0(\varkappa r)\,ds+$$

$$+\lambda_0\varkappa^3\int_L\gamma(s)\sin\alpha\cos^2\alpha_0 K_1(\varkappa r)\,ds-$$

$$-\pi(\lambda_1+\lambda_2)p(s_0)=\pi f_1(s_0),$$

$$\lambda_0\gamma''(s_0)-12h\gamma(s_0)-3(\lambda_1-\lambda_2)p(s_0)+$$

$$+\frac{3\varkappa(\lambda_1+\lambda_2)}{\pi}\int_L[\gamma'(s)\cos\alpha_0+p(s)\sin\alpha_0]K_1(\varkappa r)\,ds-$$

$$-\frac{3\varkappa^2}{\pi}(\lambda_1+\lambda_2)\int_L\gamma(s)\cos(\alpha+\alpha_0)K_0(\varkappa r)\,ds=$$

$$=f_2(s_0), \qquad (11)$$

where $\eta = k_0 + (\lambda_1 - \lambda_2)/\lambda_0$.

Setting $\lambda_1 = \lambda_2$ in (11), these equations will hold also for an open contour.† In this case, in order that the temperature and its normal derivative be continuous at the end points of the contour, the densities p(s) and γ(s) must vanish at these points.

It is noteworthy that, for $\varkappa = 0$, Eqs. (11) coincide with Eqs. (19) in [1] if the latter are somewhat transformed.

Eqs. (11) are appreciably simplified when the contact thermal conductivity $\lambda_0 = 0$—i.e., when for the intermediate layer the heat fluxes between the plate and the disk are equal and proportional to the boundary-temperature difference. Specifically, if the plate contains a linear inclusion (for example, a heat-conducting crack) instead of a disk, then for $\lambda_0 = 0$, from the first equation of the system (11), we get $p(s_0) = 0$, while the second equation of the system takes the form:

$$\frac{h}{\lambda}\gamma(s_0)-\frac{\varkappa}{2\pi}\int_L\gamma'(s)\cos\alpha_0 K_1(\varkappa r)\,ds+$$

$$+\frac{\varkappa^2}{2\pi}\int_L\gamma(s)\cos(\alpha+\alpha_0)K_0(\varkappa r)\,ds=\frac{\partial t^*}{\partial n_0}. \qquad (12)$$

In the case of a rectilinear inclusion of length $2l$, located along the Ox-axis symmetrically to the origin of the coordinates, Eq. (12) is written as follows:

$$\frac{h}{\lambda}\gamma(x)-\frac{\varkappa}{2\pi}\int_{-l}^{l}\gamma'(\xi)K_1(w)\,d\xi+$$

$$+\frac{\varkappa^2}{2\pi}\int_{-l}^{l}\gamma(\xi)K_0(w)\,d\xi=\frac{\partial t^*}{\partial y}, \qquad (13)$$

where $w = \varkappa(\xi - x)$.

Finally, we find the temperature field in a plate with a circular disk of radius R. Postulating $\alpha = \alpha_0$, $k = k_0 = 1/R$, $r = 2R\sin\alpha$ in system (11), we get

$$\varkappa(\lambda_1-\lambda_2)g_1(\varphi_0)+\varkappa\lambda_0\int_0^{2\pi}p'(\varphi)\cos\alpha K_1(\varkappa r)\,d\varphi-$$

$$-\varkappa^2R(\lambda_1-\lambda_2)g_2(\varphi_0)+$$

$$+\frac{1}{2}\lambda_0\varkappa^2\int_0^{2\pi}\gamma'(\varphi)\sin 2\alpha K_0(\varkappa r)\,d\varphi-$$

$$-\pi(\lambda_1+\lambda_2)p(\varphi_0)=\pi f_1(\varphi_0),$$

$$\frac{\lambda_0}{R^2}\gamma''(\varphi_0)-12h\dot\gamma(\varphi_0)-3(\lambda_1-\lambda_2)p(\varphi_0)+$$

$$+\frac{3\varkappa(\lambda_1+\lambda_2)}{\pi}[g_1(\varphi_0)-\varkappa Rg_2(\varphi_0)]=f_2(\varphi_0), \qquad (14)$$

where

$$g_1(\varphi_0)=\int_0^{2\pi}[\gamma'(\varphi)\cos\alpha+Rp(\varphi)\sin\alpha]K_1(\varkappa r)\,d\varphi;$$

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†This means that the plate contains a thin inclusion of a different material, distributed along the curve L.

$$g_2(\varphi_0) = \int_0^{2\pi} \gamma(\varphi) \cos 2\alpha\, K_0(\varkappa r)\, d\varphi, \quad \alpha = \frac{\varphi - \varphi_0}{2}.$$

Assume that $f_1(\varphi)$ and $f_2(\varphi)$ may be represented in the form of an absolutely converging series, as follows:

$$f_i(\varphi) = \sum_{n=0}^{\infty} f_n^{(i)}(R) \cos n\varphi \quad (i = 1,\ 2). \tag{15}$$

The solution of system (14) is taken as:

$$p(\varphi) = \sum_{n=0}^{\infty} A_n \cos n\varphi, \quad \gamma(\varphi) = \sum_{n=0}^{\infty} B_n \cos n\varphi. \tag{16}$$

We substitute (16) into (14) and evaluate the integrals, using a formula for adding the function $K_0(\varkappa r)$ [5, p. 355]. Equating the coefficients of like cosines, after certain transformations, with the aid of a relation between Bessel functions [6], we get

$$A_n = -\frac{M_n f_n^{(1)} + N_n f_n^{(2)}}{M_n Q_n + N_n p_n},$$

$$B_n = \frac{p_n f_n^{(1)} - Q_n f_n^{(2)}}{M_n Q_n + N_n p_n},$$

where

$$M_n = \lambda_0 \frac{n^2}{R} - 6(\lambda_1 + \lambda_2)\varkappa^2 R K_n' I_n' + 12h;$$

$$Q_n = 2\left[\lambda_0 \frac{n^2}{R} K_n I_n + (\lambda_1 - \lambda_2)\varkappa R K_n' I_n - \lambda_1\right];$$

$$N_n = \lambda_0 \frac{n\varkappa^2}{2}(K_{n+1} I_{n+1} - K_{n-1} I_{n-1}) + 2(\lambda_1 - \lambda_2)\varkappa^2 R K_n' I_n';$$

$$p_n = 6\left[(\lambda_1 + \lambda_2)\varkappa R K_n' I_n + \lambda_1\right].$$

Here, $\varkappa R$ is the argument of the functions K_n and I_n, while the prime denotes the derivatives of this argument.

Considering certain asymptotic expressions for the Bessel functions and treating the latter as functions of their arguments [6], for $n \to \infty$, we get

$$M_n \to \frac{\lambda_0}{R} n^2 + 9(\lambda_1 + \lambda_2)\frac{n}{R} + 12h,$$

$$N_n \to 3(\lambda_2 - \lambda_1)\frac{n}{R},$$

$$Q_n \to \frac{\lambda_0}{R} n + (3\lambda_2 - 5\lambda_1), \quad p_n \to -3(\lambda_1 + 3\lambda_2). \tag{17}$$

From the absolute convergence of series (15) and from the form of expression (17), it follows that series (16) are absolutely convergent.

Knowing $p(\varphi)$ and $\gamma(\varphi)$, from formulas (7), (8), with allowance for (6), we obtain the disk and plate temperatures:

$$t = R\sum_{n=0}^{\infty}\left[A_n K_n(\varkappa R) - \varkappa B_n K_n'(\varkappa R)\right] I_n(\varkappa\rho)\cos n\varphi$$

$$(\rho < R),$$

$$t = R\sum_{n=0}^{\infty}\left[A_n I_n(\varkappa R) - \varkappa B_n I_n'(\varkappa R)\right] K_n(\varkappa\rho)\cos n\varphi$$

$$(\rho > R). \tag{18}$$

The latter series converge uniformly. Indeed, by setting in them $\cos n\varphi = 1$, $\rho = R$, and considering that $I_n(\varkappa\rho) \le I_n(\varkappa R)$ for $\rho \le R$, and $K_n(\varkappa\rho) \le K_n(\varkappa R)$ for $\rho \ge R$, we get majorant series which, by virtue of the aforesaid conditions, converge absolutely. This confirms our previous assertion.

Let us examine the case in which a portion of the plate is immersed in a medium of constant temperature T_0 and the remaining portion, together with the disk, is exposed to a zero-temperature medium. Then

$$t_c = T_0\left[1 - S_+(y + d)\right], \tag{19}$$

where $S_+(y + d)$ is the Heaviside function; the particular solution of Eq. (1) then has the form

$$t^*(y) = \frac{T_0}{2}\left\{1 - \operatorname{sign}(y + d)\left[1 - \exp\left(-\varkappa|y + d|\right)\right]\right\}.$$

In the region $y > -d$, for $d > R$, the function $t^*(y)$ may be written in the form

$$t^*(y) = \frac{T_0}{2}\exp\left[-\varkappa(y + d)\right]$$

or in polar coordinates with an axis directed along Oy, as follows:

$$t^*(\rho, \varphi) = \frac{T_0}{2}\exp(-\varkappa d)\exp(-\varkappa\rho\cos\varphi). \tag{20}$$

Substituting (20) into formulas (4), taking into consideration that $(\partial/\partial n_0) = -(\partial/\partial\rho)$, $(\partial^2/\partial s_0^2) = (1/R^2)(\partial^2/\partial\varphi^2)$, and expanding $\exp(-\varkappa\rho\cos\varphi)$ into a series in Bessel functions [6, p. 59], after certain transformations, we get $f_1(\varphi)$ and $f_2(\varphi)$ in the form (15), where

$$f_n^{(1)} = 2(-1)^n T_0 \exp(-\varkappa d)\left[\frac{\lambda_0}{R^2} n^2 I_n + \varkappa(\lambda_1 - \lambda_2) I_n'\right];$$

$$f_n^{(2)} = 6(-1)^n \varkappa T_0(\lambda_1 + \lambda_2)\exp(-\varkappa d) I_n'.$$

The argument of function I_n is $\varkappa R$, while the prime denotes the derivative of this argument.

From known $f_n^{(1)}$ and $f_n^{(2)}$, we obtain A_n and B_n; then the temperature field is defined by expressions (18).

NOTATION

$T(x, y)$ is the temperature of plate with disk; $t_m(x, y)$ is the ambient temperature; $t^*(x, y)$ is a function characterizing the perturbation of the temperature field in the presence of the disk; $\varkappa^2 = \alpha_i/\delta\lambda_i$ (subscripts $i = 1, 2$ denote quantities pertaining to the disk and plate, respectively); α_i are the heat transfer coefficients; λ_i are the thermal conductivity coefficients; 2δ is the plate thickness; λ_0 is the reduced thermal conductivity of the intermediate layer; h is the thermal conductivity; s and s_0 are the arc coordinates of points N and M_0 on curve L; n and n_0 are the inner normal to L at points N and M_0, respectively;

$p(s)$ and $\gamma(s)$ are the density of sources and dipoles located on curve L, respectively; r is the spacing between point N on curve L and an arbitrary point of the plane xOy; $K_0(sr)$, $K_1(sr)$ are the MacDonald functions of first and second order, respectively; α and α_0 are the angles formed by vector $\overrightarrow{M_0N}$ and the positive tangents to L at points N and M_0, respectively; k and k_0 are the curvatures of L at these points; (ρ, φ) are the polar coordinates; superscripts + and − denote limiting values of quantities when approaching curve L from the side of the disk or plate, respectively.

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